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APPLICATION OF FUZZY RELATIONS TO TEST THEORY

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На відміну від класичного ймовірного підходу, в даній статті розглядається метод генерування та оцінки тестів, заснований на нечіткому підході. Це призводить до завдань, які можуть бути вирішені в рамках нечітких реляційних рівнянь. Кілька прикладів ілюструють користь такого підходу.
Ключові слова: теорія тестів, генерування та оцінка тестів, нечіткі реляційні рівняння.

В отличие от классического вероятного подхода, в данной статье рассматривается метод генерирования и оценки тестов, основанный на нечетком подходе. Это приводит к задачам, которые могут быть решены в рамках нечетких реляционных уравнений. Несколько примеров иллюстрируют пользу такого подхода.
Ключевые слова: теория тестов, генерирование и оценка тестов, нечеткие реляционные уравнения.

Unlike the classical probability-based approach we consider the ge-neration and evaluation of tests based on a fuzzy approach. This leads to tasks which can be solved within the frame of fuzzy relational equations. Several examples illustrate the usefulness of our approach.
Keywords: test theory, generation and evaluation of tests, fuzzy relational equations.

Generation of Tests with Desired Properties

Tests are one of the powerful means in modern educational systems [2]. The structure of a test is determined by items which are characterized by complexity, discrimination, correlation to the test and so on. Items are usually collected into so-called item banks that can be used for the generation of different tests. The test has to be designed from items that have desired characteristics according to test specification. The test examines the knowledge of a testee with respect to some subject, the latter being characterized by units of knowledge (UOK). Obviously, each item can be interrelated with a set of UOK.

One of the problems of test developers is the generation of a test from the item bank that has certain statistical characteristics (according to test specification) as well as a desired unit of knowledge (according to the subject that is assessed). There may be situations when it is necessary to design the tests from one subject but for different groups with different levels of knowledge (Fig.1).

The problem of choosing items is complex, because the bank of items may contain up to some thousands objects that are collected at universities or national centers of assessment. The scheme of test generation is shown in Fig.2.

Formalization of the Task and Problem Formulation

Let us consider an item bank containing N items $T = \{I_1, I_2, \ldots, I_N\}$ from some subject (e.g., mathematics). Moreover, we have M UOK $U = \{U_1, U_2, \ldots, U_M\}$ describing this subject (e.g., numbers, sets, functions, statistics, geometry,…). Let R express the quantification of the relation between the items and the UOK reflecting the fitness of the items with respect to these units:

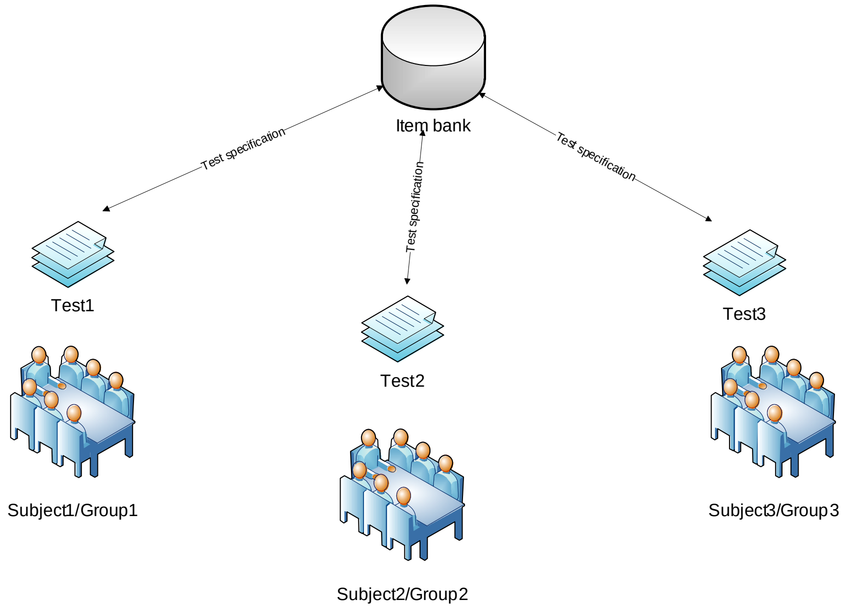

Fig. 1. Working with item bank

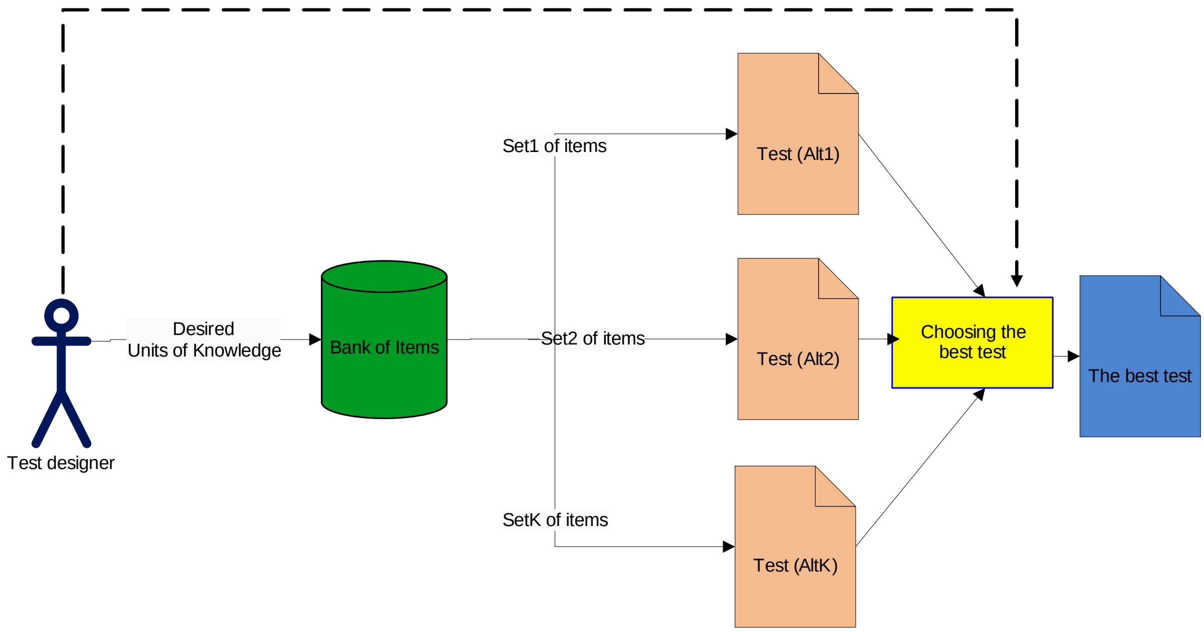

Fig. 2. Procedure of test generation

$$R = \begin{pmatrix} r_{11} & r_{12} & \dots & r_{1M} \\ r_{21} & r_{22} & \dots & r_{2M} \\ \dots & \dots & \dots & \dots \\ r_{N1} & r_{N2} & \dots & r_{NM} \end{pmatrix}.$$

The elements r_{ij} may be from the unit interval (i.e., R can be interpreted as fuzzy relation) expressing the truth degree of the fitness. Sometimes, however, it is useful to have the r_{ij} from a lattice, e.g. from set $\{0,1,\dots,S\}$. In this case the matrix elements estimate the level of correspondence of fitness. In what follows, however, we assume the unit [0,1] as basis for evaluation.

There are at least two problems to consider. First, one has to find the underlying set of UOK U^* when the testee has performed his test T^* and got the results as truth levels of answers with respect to the items. Hence, we answer the question which UOK does the testee know well. This is the *direct* problem. Second, one may be faced with the question how to choose the set of items T^* from the item bank (i.e., the test) if we want to test some subset U^* of UOK. It is clear that we may get different tests which assess the same set of UOK. This is called the *inverse* problem.

The sets T^* and U^* are supposed to be fuzzy sets on their universes *T* and *U*. The memberships are denoted by small letters and for simplicity we equate the fuzzy sets with their membership vectors, i.e. $T^* = \left(i_1^*,...,i_N^*\right)$, $U^* = \left(u_1^*,...,u_N^*\right)$.

The direct problem solution

Let $T^* = \left\{i_1^*, i_2^*, \ldots, i_N^*\right\}$ is the result of the test for some testee. Using relation *R* and T^* we can find the appropriate fuzzy set for the UOK successfully handled by the testee by computing

$$U^* = T^* \circ R \qquad (1)$$

where "$\circ$" means the max-min composition law for fuzzy relations and sets, i.e.

$$u_j^* = \max_{k \in \{1,\ldots,N\}} \min\left(i_k^*, r_{kj}\right),\ j \in \{1,\ldots,M\}. \qquad (2)$$

Example 1. Let us consider a test in mathematics containing of 10 items assessing the following units of knowledge: u_1 - Algebra, u_2 - Numbers and Expressions, u_3 - Equations and Inequalities, u_4 - Functions, u_5 - Combinatorial Calculus and Probabilities, u_6 - Statistics, u_7 - Geometry, u_8 - Plane Geometry, u_9 - Stereometry.

Moreover, we have the relation *R* (obtained from experts) between items and units of knowledge

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Assume that the testee has obtained the following result: $\left(i_1^*, i_2^*, \ldots, i_{10}^*\right) = (1,0,1,1,0,0,0,1,0,0)$. Then computation (2) yields $U^* = (1,1,1,1,0,1,1,1,0)$. It means that the testee knows u_1-u_4 and u_6-u_8, but he does not know u_5 and u_9.

Now let the answers be evaluated from a 5-degrees scale, e.g. from the set $\{0, 0.25, 0.5, 0.75, 1\}$ and suppose that the testee got the following result: $\left(i_1^*,...,i_{10}^*\right) = = (0.75, 0, 0.75, 1, 0.25, 0, 0.25, 0.5, 0.25, 0.25)$. According to (2) we find $U^* = (0.75, 0.75, 0.75, 1, 0, 0.75, 1, 1, 0.25)$. This means that the testee does not know only u_5 and knows the remaining units at different levels.

The Inverse Problem

The inverse problem consists in the determination of T^* with known *R*, U^* in (1). That is we want to know which tests might have led to the evaluation U^*. This task is much harder to solve (in comparison to the direct problem) and we may be faced with infinitely many solutions or no solution at all. It is a classical problem in the theory of fuzzy relation equations [3,5,7]. In the case of solveability the maximal (in the sense of fuzzy inclusion) solution $\hat{T} = \left(\hat{t}_1,...,\hat{t}_N\right)$ is given by

$$\hat{T} = R \alpha U^*. \qquad (3)$$

Where $\left(R \alpha U^*\right)_k = \min_{1 \le j \le M} r_{kj} \alpha u_j^*$, and the well-known α-operation (Goedel implication) is defined as

$$a \alpha b = \begin{cases} 1 \text{ for } a \le b, \\ b \text{ otherwise}. \end{cases}.$$

There may be, however, a large number of minimal solutions [7] the calculation of which is not trivial for larger *N* (typical in test theory).

Example 2. Let us consider the test with 10 items and relation from Example 1. Now we want to find the assessment of answers to items if we are given the UOK by U^* = (0.75,0.75,0.75,1,0,0.75,1,1,0.25). We obtain the maximal solution

$\hat{T}$ = (0.75,0,0.75,1,0.25,0,0.25,0.75,0.75,0.75)

and the four minimal solutions

$T_1^{\min}$ = (0.75,0,0.75,1,0,0,0,0,0,0),

$T_2^{\min}$ = (0.75,0,0,1,0,0,0,0.75,0,0),

$T_3^{\min}$ = (0,0,0.75,1,0,0,0,0,0.75,0),

$T_4^{\min}$ = (0,0,0,1,0,0,0,0.75,0.75,0).

Inverse Problems with Restrictions

Often the tester is not interested in the whole solution set of (1), but solutions with special properties are desired, as mentioned in Section 1. We distinguish two approa-ches: individual and global.

Individual Approach

In this case, we search at least one solution of (1) with *T* individual restrictions on the member values in each element I_j leading to the following task: Search T^* fulfilling

$$U^* = T^* \circ R$$

$$\underline{T} \subseteq T^* \subseteq \overline{T}, \qquad (4)$$

where $\underline{T}$, $\overline{T}$ are fuzzy sets on *T* and "$\subseteq$" means the inclusion of fuzzy sets.

This situ-ation occurs for example if we want to get a solution T^* where certain items are surpressed and other items are to be in the solution set with high evaluation.

In practice one is often faced with the problem to search for solutions with a special structure. Suppose, one has to determine a test $T = \{I_1, I_2, \ldots, I_N\}$ where item I_j takes part with probability p_j, i.e. *T* is characterized by a probability distribution *P*. This restriction can be transformed into a fuzzy set T_P^* using corresponding methods [4,6]. Due to a certain ambiguity in the choice of the transformation method and accounting that the p_j may be imprecise it seems to be more appropriate to include T_P^* in bounds, i.e. $\underline{T}_P \subseteq T_P^* \subseteq \overline{T}_P$ and we are led to task (4). The following statement enables the determination of a solution of (4) in an efficient way.

Statement 1. *Denote the solution set of (4) by Γ. Moreover let* $\tilde{T} = \check{T} \cap \overline{T}$ *with* $\check{T} = R \alpha U^*$ *(see (3)). Then* $\Gamma \neq \varnothing$ *iff* $\tilde{T} \in \Gamma$.

The proof follows from [9] where a more general situation is considered.

Example 3. Let U^* given as in Example 2. Suppose, we are interested in item soluti-ons with evaluations of at least 0.5 for items I_1, I_4, I_8, I_9. Items I_2, I_7, I_{10} are irrelevant and items I_3, I_5, I_6 should be excluded from consideration. This leads the restrictions $\underline{T} = (0.5, 0, 0, 0.5, 0, 0, 0, 0.5, 0.5, 0)$, $\overline{T} = (1, 1, 0, 1, 0, 0, 1, 1, 1, 1)$. One sees that $\tilde{T} = (0.75, 0, 0, 1, 0, 0, 0.25, 0.75, 0.75, 0.75)$ fulfills the restrictions and it is a solution, because $T_2^{\min} \subseteq \tilde{T} \subseteq \check{T}$.

Global Approach

It may be of interest to globally confine the memberships of T^* to a given (crisp) sub-set $\Omega \subseteq [0,1]$. This situation is typical when Boolean solutions are desired $(\Omega = \{0,1\})$ or solutions where the membership of each item *i* should be below a level or above another one $\left(\Omega = [0, \underline{\omega}] \cup [\overline{\omega}, 1] \text{ with } 0 \le \underline{\omega} \le \overline{\omega} \le 1\right)$. Formally this means that we search a T^* with

$$U^* = T^* \circ R, \qquad (5)$$

$$i_j^* \in \Omega \text{ for } j = 1,...,N.$$

For the analysis of (5) we apply results given in [1]. Therefore define a function $\varphi_\Omega : [0,1] \to [0,1]$ by

$$\varphi_\Omega(a) = \sup_{\substack{b \in \Omega \\ b \le a}} b. \qquad (6)$$

Remark 1.

a) For $\Omega = \{0,1\}$ (Boolean case) we obtain $\varphi_\Omega(a) = \begin{cases} 1 \text{ for } a = 1, \\ 0 \text{ otherwise}. \end{cases}$

b) For $\Omega = [0, \underline{\omega}] \cup [\overline{\omega}, 1]$ as above we have $\varphi_\Omega(a) = \begin{cases} a \text{ for } a \in \Omega, \\ \underline{\omega} \text{ otherwise}. \end{cases}$

A solution of (5) can be found by the following

Statement 2. *Denote the solution set of (5) by* Ψ_Ω *and let* Ω *be closed. Set* $\hat{T} = \varphi_\Omega(\check{T})$ *(i.e.* φ_Ω *applied elementwise). Then* $\Psi_\Omega \neq \varnothing$ *iff* $\hat{T} \in \Psi_\Omega$.

Example 4. Suppose U^* to be like in Example 2. We want to determine a solution with evaluations not lower than 0.5. Otherwise we exclude the item from further consideration. That is, $\Omega = \{0\} \cup [0.5, 1]$. A solution fulfilling the constraints is $\hat{T} = (0.75, 0, 0.75, 1, 0, 0, 0, 0.75, 0.75, 0.75)$, and obviously $T_1^{\min} \subseteq \hat{T} \subseteq \check{T}$.

Conclusion

The proposed approach of analysis and the formation of tests based on fuzzy relations opens up prospects for the automation of test generation based on the matrix elements of knowledge regarding the relationship and bank of items. Taking into account that in real test systems the item bank may contain hundreds of items, the problem of determining an optimal set of items is important. However, the demand for exact solvability may be too restrictive (i.e. Γ or Ω may be empty). Then one might search for approximative solutions (e.g. by transforming U^* into an interval-valued fuzzy set, see [8]). This will be the topic of future research.

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